Math Review Packet #2 Notes – Algebra I

I consider Algebra and algebraic thought to be the heart of mathematics—everything else before that is arithmetic. The first characteristic of algebraic thought is noticing patterns.

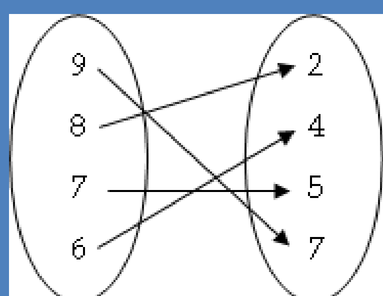
Example: Write the algebraic expression that best represents the relationship between the terms in the following sequence and n, their position in the sequence.

$$10, 6, 2, -2, -6, \ldots$$

Solution: We immediately notice that each term is decreasing by 4, so it is tempting to make our answer $n - 4$. Notice what the question asks, however: it wants the *relationship* between each term and its position, i.e. $1^{st} = 10$, $2^{nd} = 6$, $3^{rd} = 2$, etc. To find the relationship, the easiest way is to "back up" to find the "0^{th}" term (which would be our y-intercept if we were writing a linear equation), and put that together with our "slope." In our case, $10 - (-4) = 14$, so our expression becomes $14 - 4n$. You can check this by substitution.

A *relation* describes the pairing of two objects. The first set of objects is the *domain*, and the second set is the *range*. The mapping of the first set onto the second set creates *ordered pairs*.

Example: List the ordered pairs from the following mapping:



Solution: $\{(9,7),(8,2),(7,5),(6,4)\}$

In order for the relation of numbers to be a *function*, each domain value (called the independent variable) is paired with only one range value (called the dependent variable). In other words, if the same x-coordinate appears more than once with a different y-coordinate, the relation is not a function. To determine whether a graph is a function, you can also use what is called the "vertical line test"—if a vertical line touches more than one spot on a graph, then the graph is not that of a function.

Linear Functions

In Algebra I, the key function is the *linear* function. The *parent* linear function is the equation $y = x$. Every other linear function is built from this parent function, and the most often-used form is the *slope-intercept form*, $y = mx + b$, where m is the *slope* (or *rate of change* or "rise over run" or "change in y over change in x") and b is the y-intercept (where the line crosses the y-axis). To find the slope between two points (x_1, y_1) and (x_2, y_2), use the slope formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$.

If you have a slope and a point, you can write the equation of a line. There are two methods for doing this. The first uses the *point-slope form*, $y - y_1 = m(x - x_1)$, to solve for y. The second uses the slope-intercept form above to solve for b and then re-write the equation using that y-intercept.

Example: Write the equation of the line between (7, 2) and (6, 6).

Solution: First, we find the slope:

$$m = \frac{6-2}{6-7} = \frac{4}{-1} = -4$$

Method #1:

Plug in (7, 2) for (x_1, y_1) and –4 for m:	$y - 2 = -4(x - 7)$
Distribute:	$y - 2 = -4x + 28$
Solve for y:	$y = -4x + 30$

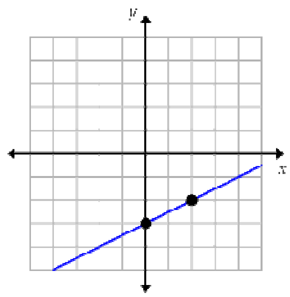
Method #2:

Plug in (7, 2) for x and y and –4 for m:	$2 = (-4)(7) + b$
Simplify:	$2 = -28 + b$
Solve for b:	$b = 30$
Plug in 30 for b:	$y = -4x + 30$

The easiest way to graph a linear function is to plot the y-intercept and then use the slope (rise over run) to plot a second point. Connect the points.

Example: Graph the equation $y = \frac{1}{2}x - 3$.

Solution: Plot the y-intercept at –3, and then go up 1 and over 2:



To write the equation of a line from a graph, determine the slope (rise over run), find the y-intercept, and plug into the slope-intercept form. If you cannot determine where the graph crosses the y-axis, you will have to select a point on the graph and use the method above to write the equation of the line.

Parallel lines have the same slope. Perpendicular lines have *opposite reciprocal* (flipped with opposite sign) slopes.

A *system* of linear equations is a set of two equations that are satisfied by the same ordered pair. There are four common methods of solving systems:

To solve a system manually:

Substitution	Solve one equation for a variable. Substitute that expression into the other equation for that variable and solve. Take your solution and substitute it *back* into the first equation to find the other variable.
Elimination	Put one or both equations into forms that, when the equations are added together, one of the terms cancels out. Solve for the remaining variable and then use substitution to solve for the other variable.

If you have a graphing calculator handy, you can also use:

Graphing	Graph the two lines ($\boxed{\text{Y=}}\boxed{\text{GRAPH}}$) and find their intersection point ($\boxed{\text{2nd}}\boxed{\text{TRACE}}\boxed{\text{5}}$).

Matrices	Set up each equation in standard form ($Ax + By = C$), and set up a 2×3 matrix on your calculator ([2nd][x^{-1}][▶][2][ENTER][3][ENTER]). In the first row, fill in the *A*, *B*, and *C* values from the first equation. In the second row, fill in the *A*, *B*, and *C* values from the second equation. Quit out of matrices ([2nd][MODE]). Use the RREF() function to calculate the solution ([2nd][x^{-1}][▶][ALPHA][APPS][2nd][x^{-1}][ENTER][ENTER]). The solution to the system is the third column of the matrix.

If there is a solution to the system, it is said to be *consistent and independent*. If the lines are parallel, then there is no solution to the system (*inconsistent*), and if the two lines are the same, it has *infinitely many solutions* (*consistent and dependent*).

<u>Linear Inequalities</u>

To graph an inequality, first treat it was if it were an equation. Draw your lines and shade as follows:

Symbol	Line	Shading
<	←- - - - - - - -→	below
>	←- - - - - - - -→	above
≤	⟷	below
≥	⟷	above

The solution to a system of inequalities is the overlap between the graphs of the inequalities.

<u>Direct Variation</u>

Direct variation is a linear equation of the form $y = kx$, where *k* is called the *constant of variation*. In direct variation, as *x* increases, *y* increases, and as *x* decreases, *y* decreases. *Indirect variation* is *not* a linear equation, but rather an equation of the form $y = \frac{k}{x}$. In indirect variation, as *x* increases, *y* decreases, and vice versa. To find the constant of variation given a point or a pair of variables, plug the values into the appropriate form for *x* and *y* and solve for *k*.

<u>Polynomials</u>

A *term* in an expression is a group of constants and/or variables that does not contain any addition or subtraction. Thus, $3x^2 - 4$ contains two terms: $3x^2$ and –4. If all of the terms in an expression have whole-number exponents, then the expression is called a *polynomial*. The *degree* of a term is the sum of the exponents in that term. The degree of a polynomial is the largest degree of any term in that polynomial. Polynomials are classified according to the number of terms and the degree of the expression:

Number of terms	one	monomial
	two	binomial
	three	trinomial
	more than three	polynomial
Degree	one	linear
	two	quadratic
	three	cubic
	four	quartic
	five	quintic

There are classifications for polynomials of degrees greater than five, but you don't come across them that often. To place a one-variable polynomial expression in *standard form*, arrange the terms in descending order by the degree of the term.

Multiplying Polynomials

To multiply two binomials, there are two methods that are usually taught, both of which are derived from the distributive property.

Method #1: FOIL

F	First	Multiply the **first** terms
O	Outer	Multiply the **outer** terms
I	Inner	Multiply the **inner** terms
L	Last	Multiply the **last** terms
		Combine like terms

Example: Multiply $(3x+2)(4x-5)$.

Solution:

F	$(3x)(4x)=12x^2$
O	$(3x)(-5)=-15x$
I	$(2)(4x)=8x$
L	$(2)(-5)=-10$
	$12x^2-15x+8x-10=12x^2-7x-10$

Method #2: Box Method

Arrange the terms in a Punnet Square and multiply.

Example: Multiply $(2x-9)(7x+3)$.

Solution: Arrange the terms as shown:

	$2x$	-9
$7x$		
3		

Multiply rows and columns:

	$2x$	-9
$7x$	$14x^2$	$-63x$
3	$6x$	-27

Combine like terms: $14x^2-63x+6x-27=14x^2-57x-27$

There are two special cases of binomial multiplication: binomial squares and difference of squares. These two are special because they have very specific patterns for their results. Binomial squares are of the form $(a\pm b)^2$, and their result is always $a^2\pm 2ab+b^2$. Differences of squares are of the form $(a+b)(a-b)$, and their result is always a^2-b^2.

Factoring polynomials in the form $ax^2 + bx + c$

This leads us into factoring. Whenever you factor a polynomial, you first want to pull out any common factors (example: $4x^2+12x$ can be factored to $4x(x+3)$). Next, check for either difference of squares or binomial squares. If neither of these patterns fits, there are three different methods you might consider:

Method #1: Factoring by Grouping

1. Multiply a and c together.
2. Find two numbers whose **product** is this number and whose **sum** is b.
3. If a = 1, the two factors will be x plus each number.
4. If a > 1, rewrite the expression using these two numbers to split b.
5. Group the expression into two pairs of binomials.
6. Factor common factors out of each group. There should be the same factor in each group.
7. Recombine the factors using the distributive property.

Example: Factor $8x^2 - 10x - 3$.

Solution:

Multiply a and c	$(8)(-3) = -24$
Find two factors	$(-12)(2) = -24$ and $-12 + 2 = -10$
Rewrite the expression	$8x^2 - 12x + 2x - 3$
Group into two pairs	$(8x^2 - 12x) + (2x - 3)$
Factor each group	$4x(2x-3) + 1(2x-3)$
Recombine	$(4x+1)(2x-3)$

Method #2: Factoring Using Boxes

This method is basically a graphical version of the method above when $a > 1$.

1. Multiply a and c.
2. Find two numbers whose product is this number and whose sum is b.
3. Fill in the boxes, using these two numbers to split b.
4. Factor each row and each column and put the factors on the outside of the boxes.
5. The outside terms are the two binomial factors.

Example: Factor $15x^2 - 2x - 8$.

Solution:

Multiply a and c	$(15)(-8) = -120$
Find two factors	$(-12)(10) = -120$ and $-12 + 10 = -2$
Fill in the boxes	$\begin{array}{\|c\|c\|} \hline 15x^2 & -12x \\ \hline 10x & -8 \\ \hline \end{array}$
Factor each row and column	$\begin{array}{c\|c\|c\|} & 5x & -4 \\ \hline 3x & 15x^2 & -12x \\ \hline 2 & 10x & -8 \\ \hline \end{array}$
Pull the outside factors	$(5x-4)(3x+2)$

Method #3: My Father Drives a Red Mustang
This is a method I teach my Algebra I students because it is straightforward and has a good mnemonic.

		Factor out any common factors.
My	Multiply	Multiply a and c.
Father	Factor	Find two factors of this product that sum to b.
Drives A	Divide	Put x plus each factor in a set of parentheses and divide the factor by a.
Red	Reduce	Reduce all fractions.
Mustang	Move	Move any denominators to the first term in front of the x.

Example: Factor $12x^2+2x-2$.

Solution:

	Factor common terms. (Don't lose this number!)	$2\left(6x^2+x-1\right)$
My	Multiply 6 and –1	$(6)(-1)=-6$
Father	Find factors	$(3)(-2)=-6$ and $3-2=1$
Drives A	Set up and divide	$2\left(x+\frac{3}{6}\right)\left(x+\frac{-2}{6}\right)$
Red	Reduce fraction(s)	$2\left(x+\frac{1}{2}\right)\left(x-\frac{1}{3}\right)$
Mustang	Move the denominator	$2(2x+1)(3x-1)$

Quadratic Functions and Equations

A quadratic function is a function of the form $y=ax^2+bx+c$ or $f(x)=ax^2+bx+c$. The "zeros" of a quadratic function are the x-intercepts of the graph (where it crosses the x-axis). The graph of a quadratic function is a parabola, whose vertex is located at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. The axis (or line) of symmetry is a vertical line at $x=\frac{-b}{2a}$.

Each coefficient in the function has a particular effect on the graph. a affects the orientation of the graph (negative opens down, positive opens up) and its width (the bigger the number, the narrower the parabola). b affects its horizontal position (positive moves it to the **left** and negative moves it to the **right**). c affects its vertical position (positive moves it up, negative moves it down).

Solving Quadratic Equations (real solutions)
To solve a quadratic function or equation, there are at least three methods you can use.

Method #1: Graphing
If you have a graphing calculator, you can graph the function and find the zeros (2nd TRACE 2 and position the "cursor" on the graph on either side of the x-axis).

The next two methods require that the equation be in standard form (and set equal to 0).

Method #2: Factoring
If the expression can be factored, set it up as a product of two binomials. Then, separate the binomials, set each binomial equal to 0, and solve for x. This is often the fastest non-calculator method.

Method #2.5: Completing the Square
I'm including this method here because while it is not my preferred method to solve a quadratic (the others are more straightforward), it always works, and it is a method we will use when we begin working with conics later on. (One of the coolest uses for this method is to derive the Quadratic Formula.)

1. Divide through by a.
2. Subtract c from both sides to put the equation in the form $x^2 \pm \frac{b}{a}x = \frac{c}{a}$
3. "Complete the square" by adding $\left(\frac{b}{2a}\right)^2$ to both sides. This turns the left side of the equation into a binomial square of the form $\left(x \pm \frac{b}{2a}\right)^2 = \frac{c}{a}$.
4. Take the square root of both sides and solve for both positive and negative roots.

Method #3: Quadratic Formula
The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, (which is derived by completing the square on the standard form) will always work to find the solutions of a quadratic equation. I encourage my students to make a list of the coefficients (a, b, and c) before they start plugging in numbers, so they can keep track of what does where. ***In this class, unless I tell you otherwise, please do not approximate irrational square roots—leave them as simplified radicals.***

Example: Solve $4x^2 + 8x - 9 = 0$.

Solution: Make a note of the coefficients (note that the sign goes with the number!).

$a = 4$, $b = 8$, $c = -9$

Substitute	$x = \dfrac{-8 \pm \sqrt{8^2 - 4(4)(-9)}}{2(4)}$
Evaluate	$x = \dfrac{-8 \pm \sqrt{64 + 144}}{8}$
Simplify	$x = \dfrac{-8 \pm \sqrt{208}}{8}$ $= \dfrac{-8 \pm 4\sqrt{13}}{8}$ $= \dfrac{-8}{8} \pm \dfrac{4\sqrt{13}}{8} = -1 \pm \dfrac{\sqrt{13}}{2}$

If we just want to know whether a quadratic equation will have imaginary roots, we can use the *discriminant* to check. This is the part of the quadratic formula that is under the radical and determines what types of solutions exist.

Discriminant	Solutions
$b^2 - 4ac > 0$	Two real solutions
$b^2 - 4ac = 0$	One real solution
$b^2 - 4ac < 0$	Two imaginary solutions

Note: If the discriminant is equal to a perfect square (1, 4, 9, etc.), the equation can be factored.